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Damped Wave Conduction and Relaxation in Cylindrical and Spherical Coordinates

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The damped wave conduction and relaxation equation was solved for infinite, cylindrical, and spherical mediums using the method of relativistic transformation of variables. The transformation $\eta = \tau^2 - X^2$ was used to transform the hyperbolic partial differential equation (which describes the wave temperature) into a Bessel differential equation in the transformation variable only. The solution is characterized by three distinct regimes. For interior points in the medium, there is a thermal inertial or zero transfer regime. The solution for the cylinder was characterized by a Bessel composite function in space and time of the half-order and first kind in the open interval $X > \tau$ and by a modified composite function in space and time of the half-order and first kind in the open interval $\tau > X$. For the sphere, the solution was characterized by a Bessel composite function in space and time of the first kind and first order in the open interval $X > \tau$ and by a modified composite function in space and time of the first order and first kind in the open interval $\tau > X$. The regimes are illustrated in figures and the solution is free of singularities.

Nomenclature

C_p	=	heat capacity, J/Kg/K
I_0	=	modified Bessel function of the zeroth order and first kind
I_1	=	modified Bessel function of the first order and first kind
$I_{1/2}$	=	modified Bessel function of the half-order and first kind
J_1	=	Bessel function of the first order and first kind
K_1	=	modified Bessel function of the first order and second kind
k	=	thermal conductivity, w/m K
q	=	heat flux, $\text{w/m}^2/\text{K}$
R	=	radius of the cylinder, sphere, m
r	=	radial coordinate, m
T	=	temperature, K
T_s	=	surface temperature, K
T_0	=	initial temperature, K
t	=	time, s
u	=	dimensionless temperature for the semi-infinite medium $(T - T_0)/(T_s - T_0)$
v_h	=	velocity of heat, $\sqrt{\alpha/\tau_r}$, m/s
w	=	wave dimensionless temperature, $u \exp(\tau/2)$
X	=	dimensionless distance, $r/(\alpha\tau_r)^{1/2}$
X_p	=	penetration distance in the infinite spherical medium, $(58.7277 + \tau_i^2)^{1/2}$, m
X_R	=	dimensional radius, $R/(\alpha\tau_r)^{1/2}$
α	=	thermal diffusivity, m^2/s
η	=	relativistic coordinate, $\tau^2 - X^2$
ρ	=	density, kg/m^3
τ	=	dimensionless time, t/τ_r
τ_{lag}	=	thermal lag time in the infinite medium, spherical coordinates, $(X_p^2 - 58.73)^{1/2}$
τ_r	=	relaxation time, s
χ	=	relativistic coordinate, $X^2 - \tau^2$

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I. Introduction

THE motivation to seek a generalized Fourier's law of heat conduction is sixfold:

1) Fourier's law of heat conduction was found to contradict the microscopic theory of reversibility introduced by Onsager [1].

2) Singularities were found in the description of transient heat conduction using the parabolic conduction equations. When transient temperature events are described using Fourier's law, a blowup occurs during short contact times in the expression for surface flux. For the cases of Sharma [2],

a) Surface flux expression in a semi-infinite body is subject to a step change in one of the boundary temperatures.

b) Surface flux for a finite slab is subject to constant wall temperature on either of its edges.

c) Temperature term in the constant wall flux problem in cylindrical coordinates in a semi-infinite medium is solved for by the Boltzmann transformation [3], leading to a solution in exponential integral.

d) In the short time limit, the parabolic conduction equations for a semi-infinite sphere are solved by using the similarity transformation.

3) Light is the speediest of velocities. Examining the solution for the transient temperature, Landau and Lifshitz [4] noted that for times greater than zero, the temperature is finite at all points in the infinite medium, except at infinite location. It can be inferred that the heat pulse has traveled at infinite speed. This is in conflict with the light speed barrier stated by the theory of relativity of Einstein. The fact that any speed of a moving object, including the thermal wave, must be less than the speed of light was examined by Kelly [5] for diffusion.

4) The development of Fourier's law of heat conduction, from observations at steady state, is of an empirical nature, but the use of it in the transient state is an extrapolation.

5) There is overprediction of theory to experiment in important industrial processes such as fluidized bed heat transfer to surfaces, chromatography, CPU overheating, adsorption, gel electrophoresis, restriction mapping, laser heating of semiconductors during manufacture of semiconductor devices, and drug delivery systems.

6) Fourier's law breaks down at small scales (Bejan [6], Casimir [7]). In this limit, the flux is described by an expression similar to the one used in radiation heat transfer (Swartz and Pohl [8]). The heat transport, for example, in dielectric crystalline materials is believed to be primarily by atomic or crystal vibrations. These vibrations

travel as waves and the energy of the waves quantitated is the phonon (Kittel [9]).

To remove the paradox in Fourier's conduction model assuming an infinite speed of propagation, the damped wave conduction and relaxation equation was originally suggested by Maxwell [10] and Morse and Feshbach [11] and postulated independently by Cattaneo [12] and Vernotte [13]. Reviews on heat waves have been presented by Joseph and Preziosi [14,15] and Osizik and Tzou [16]. A comprehensive insight into the analytical solutions using the manifestations of the generalized transport equation is given in Sharma [2]. The Cattaneo and Vernotte equation was found to be admissible within the framework of the second law of thermodynamics (Tzou [17]).

The damped wave conduction and relaxation equation is derived from the free electron theory (Sharma [18]). The relaxation time was found to be a third of the collision time between the electron and obstacle in a given material. The hyperbolic governing equation was solved for by four different methods for three different boundary conditions. The reports in the literature of a temperature overshoot was revisited. For a small slab, $a < \pi\,(\alpha\tau_r)^{1/2}$, the temperature was shown to exhibit subcritical damped oscillations. In the case of the semi-infinite medium, reports in the literature about a wave discontinuity were revisited. A substitution variable that is symmetric in space and time (i.e., $\eta = \tau^2 - X^2$) is proposed to transform the governing equation into a Bessel differential equation. Three regimes are recognized in the solution: an inertial lagging zero-transfer regime, a rising regime, and a falling regime.

Very little work has been reported in the area of analytical solutions to the damped wave conduction and relaxation in cylindrical and spherical coordinates. Barletta and Zanchini [19] attempted to obtain a solution in cylindrical coordinates using the damped wave conduction and relaxation equation and found singularities in their solution when the boundary condition was a step change. They needed to introduce a monotonic evolution of boundary temperature to remove the singularities in the solution. Brown and Churchill [20] made some experimental measurements to study the thermal waves in a cylindrical tube. In this study, the analytical solutions in cylindrical and spherical coordinates are discussed by the method of relativistic transformation of coordinates and method of separation of variables. Solutions without singularities are obtained.

II. Method of Relativistic Transformation of Coordinates in an Infinite Cylindrical Medium

Consider a fluid at an initial temperature T_0. The surface of the cylinder is maintained at a constant temperature T_s for times greater than zero. The heat propagative velocity is given as the square root of the ratio of the thermal diffusivity and relaxation time, $v_h = \sqrt{\alpha/\tau_r}$. The two time conditions, initial and final, and the two boundary conditions are

$$t = 0, \qquad r > R, \qquad T = T_0 \tag{1}$$

$$t > 0, \qquad r = R, \qquad T = T_s \tag{2}$$

$$r = \infty, \qquad t > 0, \qquad T = T_0 \tag{3}$$

The governing equation in temperature is obtained by eliminating the second cross derivative of heat flux with respect to r and t between the non-Fourier damped wave heat conduction and relaxation equation and the energy balance equation in cylindrical coordinates (Fig. 1). Considering a cylindrical shell of thickness Δr, in the limit of Δr, Δt going to zero, the energy balance equation in cylindrical coordinates becomes

$$-1/r\partial(rq_r)/\partial r = (\rho C_p)\partial T/\partial t \tag{4}$$

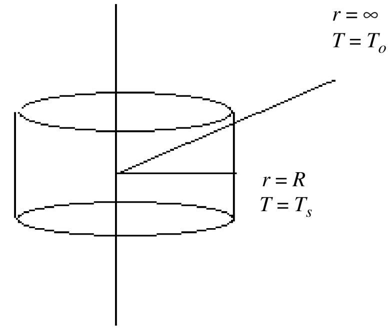

Fig. 1 Infinite medium in cylindrical coordinates heated from a cylindrical surface.

The non-Fourier damped wave equation is

$$q_r = -k\partial T/\partial r - \tau_r \partial q_r/\partial t \tag{5}$$

Multiplying Eq. (5) by r and differentiating with respect to r and then dividing by r,

$$(1/r)\partial(rq_r)/\partial r = (-k/r)\partial(r\partial T/\partial r)/\partial r - (\tau_r/r)\partial 2(rqr)/\partial t\partial r \tag{6}$$

Differentiating Eq. (4) with respect to t,

$$-1/r\partial^2(rq_r)/\partial t\partial r = (\rho C_p)\partial^2 T/\partial t^2 \tag{7}$$

Substituting Eqs. (4) and (7) into Eq. (6), the governing equation in temperature is given as

$$(\rho C_p \tau_r)\partial^2 T/\partial t^2 + (\rho C_p)\partial T/\partial t = (k/r)\partial(r\partial T/\partial r)/\partial r \tag{8}$$

obtaining the dimensionless variables

$$u = (T - T_0)/(T_s - T_0), \qquad \tau = (t/\tau_r), \qquad X = r/\sqrt{\alpha\tau_r} \tag{9}$$

The governing equation in the dimensionless form can be written as

$$\partial u/\partial\tau + \partial^2 u/\partial\tau^2 = \partial^2 u/\partial X^2 + 1/X\partial u/\partial X \tag{10}$$

The damping term is removed from the governing equation, which is done realizing that the transient temperature decays with time in an exponential fashion. The other reason for this maneuver is to study the wave equation without the damping term. Let $u = w\exp(-\tau/2)$, and the damping component of the equation is removed to yield

$$-w/4 + \partial^2 w/\partial\tau^2 = \partial^2 w/\partial X^2 + 1/X\partial w/\partial X \tag{11}$$

The solution for the wave temperature determines the different regimes of transfer. The wave temperature given by Eq. (11) can be solved by using the method of relativistic transformation of coordinates. Consider the transformation variable [21–24] η as

$$\eta = \tau^2 - X^2 \quad \text{for } \tau > X \tag{12}$$

The governing equation (11) becomes

$$(\partial^2 w/\partial\eta^2)4(\tau^2 - X^2) + 6(\partial w/\partial\eta) - w/4 = 0 \tag{13}$$

$$4\eta^2\partial^2 w/\partial\eta^2 + 6\eta\partial w/\partial\eta - \eta w/4 = 0 \tag{14}$$

or

$$\eta^2\partial^2 w/\partial\eta^2 + 3/2\eta\partial w/\partial\eta - \eta w/16 = 0 \tag{15}$$

Comparing Eq. (15) with the generalized Bessel equation [2], the solution is $a = 3/2$, $b = 0$, $c = 0$, $d = -1/16$, and $s = 1/2$. The order p of the solution is then $p = 2\sqrt{1/16} = 1/2$ and

$$w = \frac{c_1 I_{1/2}(1/2\sqrt{(\tau^2 - X^2)})}{(\tau^2 - X^2)^{1/4}} + \frac{c_2 I_{-1/2}(1/2\sqrt{(\tau^2 - X^2)}}{(\tau^2 - X^2)^{1/4}} \tag{16}$$

where c_2 can be seen to be zero, because w is finite and not infinitely large at $\eta = 0$. Thus,

$$w = \frac{c_1 I_{1/2}(1/2\sqrt{(\tau^2 - X^2)})}{(\tau^2 - X^2)^{1/4}} \quad \text{for } \tau > X \tag{17}$$

Thus, the wave temperature is given by a modified Bessel composite function in space and time of the half-order and first kind divided by the quartic root of the transformation variable η. For $X > \tau$, let $\chi = X^2 - \tau^2$. The governing equation (11) becomes

$$(\partial^2 w/\partial \eta^2)4(X^2 - \tau^2) + 6(\partial w/\partial \eta) + w/4 = 0 \tag{18}$$

$$4\chi^2 \partial^2 w/\partial \eta^2 + 6\chi \partial w/\partial \eta - \chi w/4 = 0 \tag{19}$$

or

$$\chi^2 \partial^2 w/\partial \eta^2 + 3/2\chi \partial w/\partial \eta + \chi w/16 = 0 \tag{20}$$

Comparing Eq. (20) with the generalized Bessel equation [2], the solution is $a = 3/2$, $b = 0$, $c = 0$, $d = +1/16$, and $s = 1/2$. The order p of the solution is then $p = 2\sqrt{1/16} = 1/2$ and

$$w = \frac{c_1 J_{1/2}(1/2\sqrt{(X^2 - \tau^2)})}{(X^2 - \tau^2)^{1/4}} + \frac{c_2 J_{-1/2}(1/2\sqrt{(X^2 - \tau^2)})}{(X^2 - \tau^2)^{1/4}} \tag{21}$$

where c_2 can be seen to be zero, because w is finite and not infinitely large at $\chi = 0$. Thus, the solution for the wave temperature w will become a Bessel composite function in space and time of the half-order and first kind divided by the quartic root of the transformation variable χ for $X > \tau$. Thus,

$$w = \frac{c_1 J_{1/2}(1/2\sqrt{(X^2 - \tau^2)})}{(X^2 - \tau^2)^{1/4}} \quad \text{for } X > \tau \tag{22}$$

It can be seen that when the argument in the Bessel function reaches $4\pi^2$, the wave temperature given by Eq. (22) will become zero. Beyond this point, the wave temperature will be zero. From this finding, the thermal inertial lag time and penetration length will later be calculated for a given point in the infinite medium or given instant in time. Thus, three distinct regimes of the solution can be seen. A first inertial regime, a second Bessel composite function regime, and a third modified composite function regime. The boundary conditions can be solved for in a number of ways from this point. One method is to determine the boundary condition by considering what happens at the wave front: the transformation variable η is zero. Hence, Eq. (13) becomes

$$\partial w/\partial \eta = w/24 \tag{23}$$

Integrating both sides of Eq. (23) for $\tau > X$,

$$\ln(w) = \eta/24 + c' \quad \text{or} \quad w = c'' \exp(\eta/24) \tag{24}$$

At the wave front,

$$u = c'' \exp(-\tau/2) = c'' \exp(-X/2) \tag{25}$$

From the boundary condition given by Eq. (2),

$$c'' = \exp(X_R/2) \tag{26}$$

where $X_R = R/(\alpha\tau_r)^{1/2}$. Thus,

$$u = \frac{\exp[-(\tau - X_R)/2]I_{1/2}1/2\sqrt{(\tau^2 - X^2)}}{(\tau^2 - X^2)^{1/4}} \quad \text{for } \tau > X \tag{27}$$

$$u = \frac{\exp[-(\tau - X_R)/2]J_{1/2}1/2\sqrt{(X^2 - \tau^2)}}{(X^2 - \tau^2)^{1/4}} \quad \text{for } X > \tau \tag{28}$$

Another way to solve for the boundary condition is to seek an approximate solution by eliminating c_1 between Eq. (17) and the equation that describes the boundary condition. It can be noted that this is a mild function of time, however, because the general solution of PDE consists of n arbitrary functions when the order of the PDE is n, compared with n arbitrary constants for the ODE. Thus, the second-order hyperbolic PDE, when solved for using a transformation variable, will have two arbitrary functions to be solved for. From the boundary condition at $X = X_R$,

$$1 = \frac{\exp(-\tau/2)c_1 I_{1/2}(1/2\sqrt{\left(\tau^2 - X_R^2\right)}}{\left(\tau^2 - X_R^2\right)^{1/4}} \tag{29}$$

Thus, the temperature in the infinite medium can be given by

$$u = \left[\left(\tau^2 - X_R^2\right)^{1/4}/(\tau^2 - X^2)^{1/4}\right] \times \left[I_{1/2}1/2\sqrt{\tau^2 - X^2}\Big/ I_{1/2}1/2\sqrt{\tau^2 - X_R^2}\right] \tag{30}$$

In terms of elementary functions, Eq. (30) can be written as

$$u = \left[\left(\tau^2 - X_R^2\right)^{1/2}/(\tau^2 - X^2)^{1/2}\right] \times \left[\sinh 1/2\sqrt{\tau^2 - X^2}\Big/ \sinh 1/2\sqrt{\tau^2 - X_R^2}\right] \tag{31}$$

In the limit of X_R going to zero, the expression becomes

$$u = [\tau/(\tau^2 - X^2)^{1/2}]\left[\sinh 1/2\sqrt{\tau^2 - X^2}\Big/ \sinh(\tau/2)\right] \quad \text{for } \tau > X \tag{32}$$

In a similar fashion for $X > \tau$,

$$u = \left[\left(X_R^2 - \tau^2\right)^{1/4}/(X^2 - \tau^2)^{1/4}\right] \times \left[J_{1/2}1/2\sqrt{X^2 - \tau^2}\Big/ J_{1/2}1/2\sqrt{X_R^2 - \tau^2}\right] \tag{33}$$

Equation (33) can be written in terms of trigonometric functions as

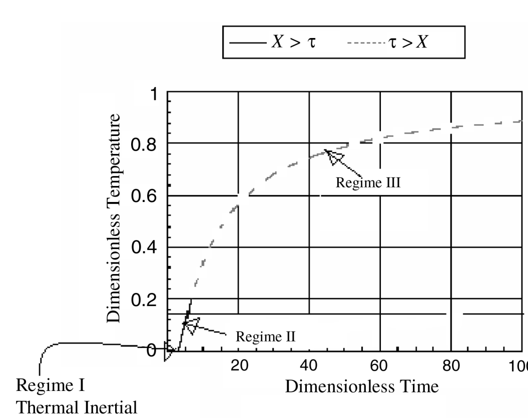

Fig. 2 Three regimes of dimensionless temperature at $X_p = 7$ in the infinite cylindrical medium.

$$u = \left[\left(X_R^2 - \tau^2\right)^{1/2} / (X^2 - \tau^2)^{1/2}\right] \times \left[\sin 1/2\sqrt{X^2 - \tau^2} \Big/ \sin 1/2\sqrt{X_R^2 - \tau^2}\right] \tag{34}$$

The dimensionless temperature at a point in the medium (e.g., $X_p = 7$) is considered and shown in Fig. 2. Three different regimes can be seen. The first regime is that of the thermal lag and consists of no change from the initial temperature. The second regime is when

$$\tau_{\text{lag}}^2 = X^2 - 4\pi^2 \quad \text{or } \tau_{\text{lag}} = \sqrt{X_p^2 - 4\pi^2} = 3.09 \quad \text{when } X_p = 7 \tag{35}$$

For times greater than the time lag and less than X_p, the dimensionless temperature is given by Eq. (34). For dimensionless times greater than seven, the dimensionless temperature is given by Eq. (32). For distances closer to the surface compared with 2π, the time lag will be zero.

III. Relativistic Transformation of Spherical Coordinates in an Infinite Medium

Consider a fluid at an initial temperature T_0. The surface of a solid sphere is maintained at a constant temperature T_s for times greater than zero (Fig. 3). The heat propagative velocity is given as the square root of the ratio of the thermal diffusivity and relaxation time , $v_h = \sqrt{\alpha/\tau_r}$. The two time conditions, initial and final, and the two boundary conditions are

$$t = 0, \qquad r > R, \qquad T = T_0 \tag{36}$$

$$t = \infty, \qquad T = T_s \quad \forall R \tag{37}$$

$$t > 0, \qquad r = R, \qquad T = T_s \tag{38}$$

$$r = \infty, \qquad t > 0, \qquad T = T_0 \tag{39}$$

The governing equation in temperature is obtained by eliminating the second cross derivative of heat flux with respect to r and t between the non-Fourier damped wave heat conduction and relaxation equation and the energy balance equation in spherical coordinates, and is given by Eq. (40):

$$(\rho C_p \tau_r)\partial^2 T/\partial t^2 + (\rho C_p)\partial T/\partial t = (k/r^2)\partial(r^2 \partial T/\partial r)/\partial r \tag{40}$$

obtaining the dimensionless form of the governing equation

$$\partial u/\partial\tau + \partial^2 u/\partial\tau^2 = \partial^2 u/\partial X^2 + 2/X\partial u/\partial X \tag{41}$$

where

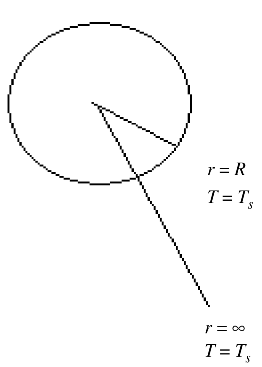

Fig. 3 Semi-infinite medium heated from a solid spherical surface.

$$u = (T - T_0)/(T_s - T_0), \qquad \tau = (t/\tau_r), \qquad X = r/\sqrt{\alpha\tau_r} \tag{42}$$

The damping term is removed from the governing equation, which is done realizing that the transient temperature decays with time in an exponential fashion. The other reason for this maneuver is to study the wave equation without the damping term. Let $u = w\exp(-\tau/2)$ and

$$-w/4 + \partial^2 w/\partial\tau^2 = \partial^2 w/\partial X^2 + 2/X\partial w/\partial X \tag{43}$$

Equation (43) can be solved by using the method of relativistic transformation of coordinates. Consider the transformation variable η as

$$\eta = \tau^2 - X^2 \quad \text{for } \tau > X \tag{44}$$

Then, writing Eq. (43) in terms of the transformation variable,

$$(\partial^2 w/\partial\eta^2)4(\tau^2 - X^2) + 8(\partial w/\partial\eta) - w/4 = 0 \tag{45}$$

$$4\eta^2\partial^2 w/\partial\eta^2 + 8\eta\partial w/\partial\eta - \eta w/4 = 0 \tag{46}$$

$$\eta^2\partial^2 w/\partial\eta^2 + 2\eta\partial w/\partial\eta - \eta w/16 = 0 \tag{47}$$

Comparing Eq. (47) with the generalized Bessel equation given by Sharma [2], the solution is $a = 2$, $b = 0$, $c = 0$, $s = 1/2$, and $d = -1/16$. The order of the Bessel solution would be $p = 2\sqrt{1/4} = 1$ then $\sqrt{|d|/s} = 1/2$. Hence, the solution to Eq. (47) can be written as

$$w = \frac{c_1 I_1(1/2\sqrt{(\tau^2 - X^2)}}{(\tau^2 - X^2)^{1/2}} + \frac{c_2 K_1(1/2\sqrt{(\tau^2 - X^2)}}{(\tau^2 - X^2)^{1/2}} \tag{48}$$

where c_2 can be seen to be zero, because w is finite and not infinitely large at $\eta = 0$. The solution is in terms of a composite modified Bessel function of the first order and first kind. Therefore, the temperature can be written as

$$u = \frac{c_1\exp(-\tau/2)I_1(1/2\sqrt{(\tau^2 - X^2)})}{(\tau^2 - X^2)^{1/2}} \tag{49}$$

From the boundary condition at the solid surface,

$$1 = \frac{c_1\exp(-\tau/2)I_{1/2}\left(1/2\sqrt{\left(\tau^2 - X_R^2\right)}\right)}{\left(\tau^2 - X_R^2\right)^{1/2}} \tag{50}$$

Dividing Eq. (49) by Eq. (50), the solution for u can be given in a more usable form for $\tau > X$:

$$u = \left[\left(\tau^2 - X_R^2\right)^{1/2} / (\tau^2 - X^2)^{1/2}\right] \times \left[I_{1/2} 1/2\sqrt{\tau^2 - X^2} \Big/ I_{1/2} 1/2\sqrt{\tau^2 - X_R^2}\right] \tag{51}$$

For $X > \tau$, let $\chi = X^2 - \tau^2$. Then, writing Eq. (43) in terms of the transformation variable,

$$(\partial^2 w/\partial\eta^2)4(X^2 - \tau^2) + 8(\partial w/\partial\eta) + w/4 = 0 \tag{52}$$

$$4\chi^2\partial^2 w/\partial\eta^2 + 8\chi\partial w/\partial\eta + \chi w/4 = 0 \tag{53}$$

or

$$\chi^2\partial^2 w/\partial\eta^2 + 2\chi\partial w/\partial\eta + \chi w/16 = 0 \tag{54}$$

Comparing Eq. (54) with the generalized Bessel equation as given in Sharma [2], the solution is $a = 2$, $b = 0$, $c = 0$, $s = 1/2$, and $d = +1/16$. The order of the Bessel solution would be $p = 2\sqrt{1/4} = 1$ and $\sqrt{|d|/s} = 1/2$. Hence, the solution to Eq. (54) can be written as

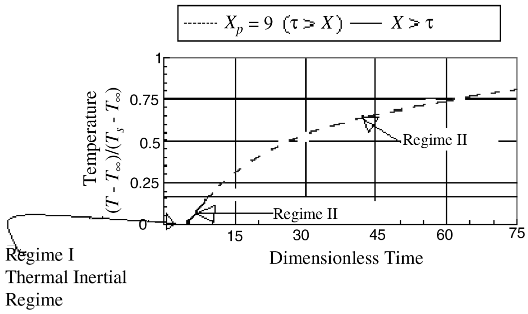

Fig. 4 Three regimes of dimensionless temperature in the exterior point in an infinite sphere at $X_p = 9$.

$$w = \frac{c_1 J_1(1/2\sqrt{(X^2-\tau^2)}}{(X^2-\tau^2)^{1/2}} + \frac{c_2 Y_1(1/2\sqrt{(X^2-\tau^2)}}{(X^2-\tau^2)^{1/2}} \tag{55}$$

where c_2 can be seen to be zero, because w is finite and not infinitely large at $\eta = 0$. The solution is in terms of a composite modified Bessel function of the first order and first kind. Therefore, the temperature can be written as

$$u = \frac{c_1 \exp(-\tau/2) J_1(1/2\sqrt{(\tau^2-X^2)}}{(X^2-\tau^2)^{1/2}} \tag{56}$$

From the boundary condition at the solid surface,

$$1 = \frac{c_1 \exp(-\tau/2) J_1\left(1/2\sqrt{\left(X_R^2-\tau^2\right)}\right.}{\left(X_R^2-\tau^2\right)^{1/2}} \tag{57}$$

Dividing Eq. (56) by Eq. (57), the solution for u can be given in a more usable form for $X > \tau$:

$$u = \left[\left(X_R^2-\tau^2\right)^{1/2} / (X^2-\tau^2)^{1/2}\right] \times \left[J_1 1/2\sqrt{X^2-\tau^2} \Big/ J_1 1/2\sqrt{X_R-\tau^2}\right] \tag{58}$$

Equation (58) can be written for $X > \tau$. For $X = \tau$, the solution at the wave front result can be obtained by solving Eq. (43) at $\eta = 0$. In the limit of X_R going to zero for $\tau > X$,

$$u = [\tau/(X^2-\tau^2)^{1/2}]\left[I_1 1/2\sqrt{\tau^2-X^2} \Big/ I_1(\tau/2)\right] \tag{59}$$

Seventeen terms were taken in the series expansion of the modified Bessel composite function of the first kind and first order and the Bessel composite function of the first kind and first order, respectively, and the results are plotted in Fig. 4 for a given $X_p = 9$, using a Microsoft Excel spreadsheet on a Pentium IV desktop microcomputer. Three regimes can be identified. The first regime is that of the thermal lag and consists of no change from the initial temperature. The second regime is when

$$\tau_{\text{lag}}^2 = X^2 - (7.6634)^2 \quad \text{or} \quad \tau_{\text{lag}} = \sqrt{X_p^2 - 7.6634^2} = 4.72 \quad \text{when } X_p = 9 \tag{60}$$

The first zero of $J_1(x)$ occurs at $x = 3.8317$. The value of 7.6634 is twice the first root of the Bessel function of the first order and first kind. For times greater than the time lag and less than X_p, the dimensionless temperature is given by Eq. (58). For dimensionless times greater than seven, the dimensionless temperature is given by Eq. (59). For distances closer to the surface than with $7.6634\sqrt{\alpha\tau_r}$, the thermal lag time will be zero. The ballistic term manifests as a thermal lag at a given point in the medium.

IV. Conclusions

Earlier attempts by other investigators to obtain an analytical solution for the damped wave conduction and relaxation equation in an infinite cylindrical medium were made by using the method of Laplace transforms. They found singularities in the results for temperature for a step change in temperature. In this study, the dimensionless temperature in an infinite cylindrical medium using the damped wave conduction and relaxation is obtained by the method of relativistic transformation of coordinates. The transformation $\eta = (\tau^2 - X^2)$ was found to transform the governing equation for the wave temperature (temperature, less the damping component) to a Bessel differential equation in the transformation variable for both the infinite cylinder and sphere. The solution to the infinite cylinder was characterized by a modified Bessel composite function in space and time of the first kind and half-order in the open interval of $\tau > X$. This is when the wave speed ($\sim r/t$) is smaller than the diffusion speed $(\alpha/\tau_r)^{1/2}$. For values of times less than X, the solution was characterized by a Bessel composite function in space and time of the first kind and half-order. This is when the wave speed is greater than the diffusion speed. The inertial time lagging regime marked the three distinct regimes of transfer. This is when the wave speed and diffusion speed are equal to each other. For the infinite sphere, the solutions were characterized by a modified Bessel composite function in space and time of the first kind and first order and by a Bessel composite function in space and time of the first kind and first order for the open intervals of $\tau > X$ and $X > \tau$. The initial condition can be verified in the asymptotic limits of zero time from the governing equation. The transformation variable is symmetric in space and time.

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